Signature matrices of membranes

Leonard Schmitz (TU Berlin)

joint w/ **Felix Lotter** (MPI MiS Leipzig)

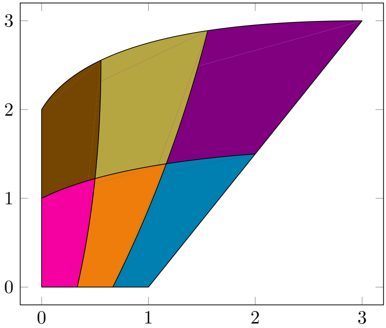
https://arxiv.org/abs/2409.11996

Oxford-Berlin meeting

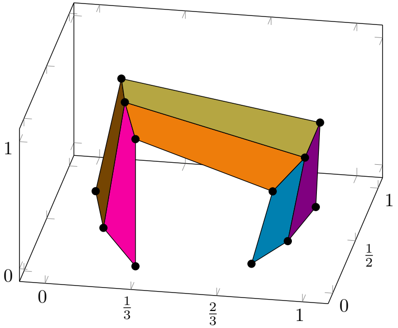
2024-12-10

supported by CRC/TRR 388 Project A4

Signatures for membranes $[0,1]^2 \to \mathbb{R}^d$



$$(s,t) \mapsto \begin{pmatrix} st - st^2 + s^2t + s^2t^2 + s \\ st + st^2 - s^2t^2 + 2t \end{pmatrix}$$



A piecewise bilinear interpolation of 12 points in $\mathbb{R}^3$ on a 4×3 grid

[1] **C. Giusti, D. Lee, V. Nanda, H. Oberhauser**. *"A topological approach to mapping space signatures"*. **2022**

[2] **J. Diehl, K. Ebrahimi-Fard, F. Harang, S. Tindel**, *"On the signature of an image"*. **2024**

Signature matrices

$\partial_{12}X(s,t) := \frac{\partial s \partial t}{\partial^2} X(s,t)$

For $X : [0,1]^2 \to \mathbb{R}^d$ we define $\sigma = \sigma(X) \in \mathbb{R}^{d\times d}$ via

$$\sigma_{i,j} := \int_0^1 \int_0^1 \int_0^{t_2} \int_0^{s_2} \partial_{12}X_i(s_1,t_1)\, \partial_{12}X_j(s_2,t_2)\, \mathrm{d}s_1 \mathrm{d}t_1 \mathrm{d}s_2 \mathrm{d}t_2$$

This matrix is known as the *2nd level* of the *id-signature* of X

Example. For $d = 2$, the *bilinear membrane* $X(s,t) := \begin{pmatrix} 2\,st \\ 6\,st \end{pmatrix}$ yields

$$\sigma = \frac{1}{2^2} \begin{pmatrix} 2 \\ 6 \end{pmatrix} \begin{pmatrix} 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

Signature varieties for membranes

Main result. "The entries of the signature matrix $\sigma = \sigma(X) \in \mathbb{R}^{d \times d}$ satisfy in general <u>no</u> algebraic relations."

To show this, we enter the realm of [3].

Remark. For *signature matrices of paths* this is not true, e.g.

$$\begin{aligned} 4\sigma_{1,1}\sigma_{2,2} &= \sigma_1^2\sigma_2^2 = (\sigma_1\sigma_2)^2 \\ &= (\sigma_{1,2} + \sigma_{2,1})^2 \\ &= \sigma_{2,1}^2 + 2\sigma_{2,1}\sigma_{1,2} + \sigma_{1,2}^2 \end{aligned}$$

for all σ due to shuffle relations.

[3] **C. Améndola**, **P. Friz**, **B. Sturmfels**, *"Varieties of signature tensors"*. **2019**

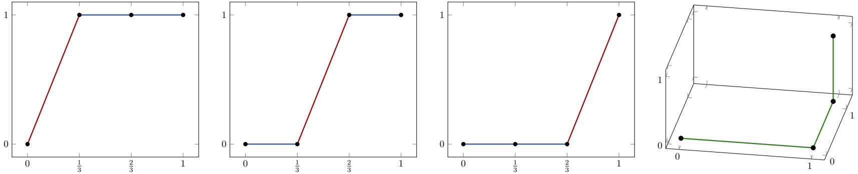
Dictionaries for paths

$\mathrm{Mom}^m : [0,1] \to \mathbb{R}^m,\ t \mapsto (t, t^2, \ldots, t^m)$ is a *dictionary* for

$$\left\{ X : [0,1] \to \mathbb{R}^d \;\middle|\; \begin{array}{l} X_j \text{ polynomial} \\ \deg(X_j) \le m \\ X_j(0) = 0 \end{array} \right\} = \left\{ A\,\mathrm{Mom}^m \mid A \in \mathbb{R}^{d \times m} \right\}$$

For its signature matrix $\sigma(\mathrm{Mom}^m)_{i,j} = \frac{j}{(i+j)}$

Similarly Axis^m is a dictionary for m-segments paths



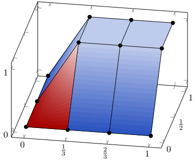
$\mathrm{graph}(\mathrm{Axis}^3_1)$ $\mathrm{graph}(\mathrm{Axis}^3_2)$ $\mathrm{graph}(\mathrm{Axis}^3_3)$ $\mathrm{Im}(\mathrm{Axis}^3)$

[4] **M. Pfeffer**, **A. Seigal**, **B. Sturmfels**, *"Learning paths from signature tensors"*. **2019**

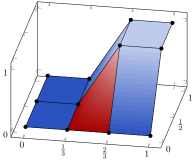
Dictionaries for membranes

For *polynomial membranes*, $\mathrm{Mom}^{m,n}(s,t) := \mathrm{Mom}^{m}(s) \otimes \mathrm{Mom}^{n}(t)$

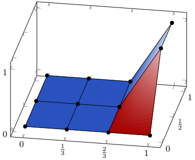
and *piecewise bilinear*, $\mathrm{Axis}^{m,n}(s,t) := \mathrm{Axis}^{m}(s) \otimes \mathrm{Axis}^{n}(t)$



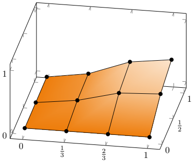
$\mathrm{Axis}^{3,2}_{1,1}$



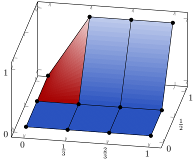
$\mathrm{Axis}^{3,2}_{2,1}$



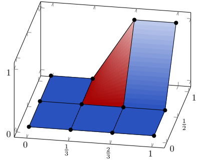
$\mathrm{Axis}^{3,2}_{3,1}$



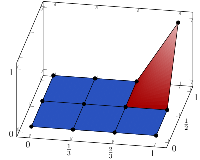
$A\,\mathrm{Axis}^{3,2}$



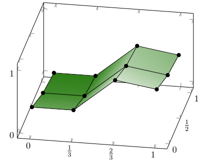
$\mathrm{Axis}^{3,2}_{1,2}$



$\mathrm{Axis}^{3,2}_{2,2}$



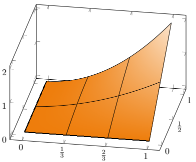
$\mathrm{Axis}^{3,2}_{3,2}$



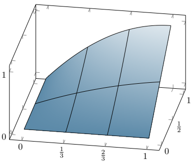
R

Decomposition of 1-dim piecewise bilinear membrane $X = A\,\mathrm{Axis}^{m,n} + R$

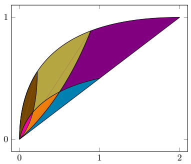
Piecewise and polynomial membranes



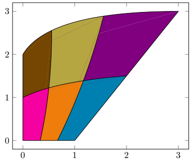
graph(X_1)



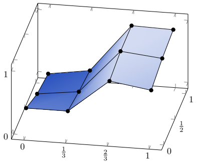
graph(X_2)



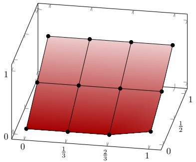
Im(X)



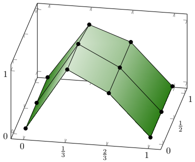
Im($X + \binom{s}{2t}$)



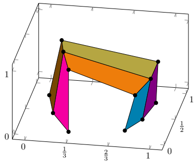
graph(Y_1)



graph(Y_2)



graph(Y_3)



Im(Y)

A polynomial membrane $X(s,t) = \begin{pmatrix} st - st^2 + s^2t + s^2t^2 \\ st + st^2 - s^2t^2 \end{pmatrix}$ and a piecewise bilinear membrane Y interpolating 12 points in $\mathbb{R}^3$.

Signature as an algebraic map

Equivariance: for all $Y : [0,1]^2 \to \mathbb{R}^{\mu}$ and $A \in \mathbb{R}^{d\times\mu}$,

$$\sigma(A\,Y) = A\,\sigma(Y)\,A^{\top}$$

Theorem [Lotter, S]**.**

$$\left\{ \sigma(X) \;\middle|\; \begin{array}{l} X : [0,1]^2 \to \mathbb{R}^d \\ \text{polynomial} \\ \text{with multi-degree} \leq (m,n) \end{array} \right\} = \left\{ \sigma(X) \;\middle|\; \begin{array}{l} X : [0,1]^2 \to \mathbb{R}^d \\ \text{piecewise bilinear} \\ \text{of order } \leq (m,n) \end{array} \right\}$$

Proof. Equivariance, $\sigma(\mathrm{Mom}^{m,n}) = \sigma(\mathrm{Mom}^{m}) \otimes \sigma(\mathrm{Mom}^{n})$ and [3].

Definition. Let $\mathcal{M}_{d,(m,n)} := \overline{\mathrm{Im}(\phi)}$ be the *signature variety* where $\phi : \mathbb{C}^{d\times mn} \to \mathbb{C}^{d^2}, A \mapsto A\,\sigma(\mathrm{Mom}^{m,n})\,A^{\top}$

[3] **C. Améndola**, **P. Friz**, **B. Sturmfels**, *"Varieties of signature tensors"*. **2019**

Example.

For any polynomial membrane

$$X = A\,\mathrm{Mom}^{2,2} = \begin{pmatrix} a_{1,1}st + a_{1,2}st^2 + a_{1,3}s^2t + a_{1,4}s^2t^2 \\ a_{2,1}st + a_{2,2}st^2 + a_{2,3}s^2t + a_{2,4}s^2t^2 \end{pmatrix}$$

we obtain $\sigma(X) = A\,\sigma(\mathrm{Mom}^{2,2})\,A^\top$, e.g. with the *dictionary* $\mathrm{Mom}^{2,2}(s,t) = (st, st^2, s^2t, s^2t^2)$ and its *core matrix*

$$\sigma(\mathrm{Mom}^{2,2}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{4}{9} \\ \frac{1}{6} & \frac{1}{4} & \frac{2}{9} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

the homogeneous polynomial

$$\begin{aligned} \sigma(X)_{1,1} = \frac{1}{4}a_{1,1}^2 + \frac{1}{2}a_{1,1}a_{1,2} + \frac{1}{2}a_{1,1}a_{1,3} + \frac{5}{9}a_{1,1}a_{1,4} + \frac{1}{4}a_{1,2}^2 \\ + \frac{4}{9}a_{1,2}a_{1,3} + \frac{1}{2}a_{1,2}a_{1,4} + \frac{1}{4}a_{1,3}^2 + \frac{1}{2}a_{1,3}a_{1,4} + \frac{1}{4}a_{1,4}^2 \end{aligned}$$

Varieties of signature matrices

We obtain a stabilizing grid

$$
\begin{array}{ccccccccc}
\mathcal{M}_{d,(1,1)} & \subseteq & \mathcal{M}_{d,(1,2)} & \subseteq & \mathcal{M}_{d,(1,3)} & \subseteq & \dots & \subseteq & \mathcal{M}_{d,(1,N)} \\
\cap\!| & & \cap\!| & & \cap\!| & & & & \cap\!| \\
\mathcal{M}_{d,(2,1)} & \subseteq & \mathcal{M}_{d,(2,2)} & \subseteq & \mathcal{M}_{d,(2,3)} & \subseteq & \dots & \subseteq & \mathcal{M}_{d,(2,N)} \\
\cap\!| & & \cap\!| & & \cap\!| & & & & \cap\!| \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots \\
\cap\!| & & \cap\!| & & \cap\!| & & & & \cap\!| \\
\mathcal{M}_{d,(M,1)} & \subseteq & \mathcal{M}_{d,(M,2)} & \subseteq & \mathcal{M}_{d,(M,3)} & \subseteq & \dots & \subseteq & \mathcal{M}_{d,(M,N)}
\end{array}
$$

where $M, N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,

$$
\mathcal{M}_{d,(m,1)} \subseteq \dots \subseteq \mathcal{M}_{d,(m,N-1)} \subseteq \mathcal{M}_{d,(m,N)} = \mathcal{M}_{d,(m,N+1)} = \dots
$$
$$
\mathcal{M}_{d,(1,n)} \subseteq \dots \subseteq \mathcal{M}_{d,(M-1,n)} \subseteq \mathcal{M}_{d,(M,n)} = \mathcal{M}_{d,(M+1,n)} = \dots
$$

Theorem [Lotter, S]. i) $\mathcal{M}_{d,(M,N)} = \mathbb{C}^{d^2}$
ii) $\mathcal{M}_{d,(1,N)} \cong \mathcal{M}_{d,(M,1)}$ *universal path variety*

Example.

i) For $d = 2$ and $(m,n) = (2,1)$ we have $A\,\sigma(\mathrm{Mom}^{(2,1)})\,A^\top =$

$$\begin{pmatrix} \frac{1}{4}a_{1,1}^2 + \frac{1}{2}a_{1,1}a_{1,2} + \frac{1}{4}a_{1,2}^2 & \frac{1}{4}a_{1,1}a_{2,1} + \frac{1}{3}a_{1,1}a_{2,2} + \frac{1}{6}a_{2,1}a_{1,2} + \frac{1}{4}a_{1,2}a_{2,2} \\ \frac{1}{4}a_{1,1}a_{2,1} + \frac{1}{6}a_{1,1}a_{2,2} + \frac{1}{3}a_{2,1}a_{1,2} + \frac{1}{4}a_{1,2}a_{2,2} & \frac{1}{4}a_{2,1}^2 + \frac{1}{2}a_{2,1}a_{2,2} + \frac{1}{4}a_{2,2}^2 \end{pmatrix}$$

and with *Gröbner bases* we can show,

$$\mathcal{M}_{2,(2,1)} = \mathcal{V}(4\sigma_{1,1}\sigma_{2,2} - \sigma_{2,1}^2 - 2\sigma_{2,1}\sigma_{1,2} - \sigma_{1,2}^2)$$

ii) For $(m,n) = (2,2)$ the entries of $S = A\,\sigma(\mathrm{Mom}^{(2,2)})\,A^\top$ are

$$\begin{aligned} S_{2,1} = {} & \frac{1}{4}a_{1,1}a_{2,1} + \frac{1}{6}a_{1,1}a_{2,2} + \frac{1}{6}a_{1,1}a_{2,3} + \frac{1}{9}a_{1,1}a_{2,4} + \frac{1}{3}a_{2,1}a_{1,2} + \frac{1}{3}a_{2,1}a_{1,3} + \frac{4}{9}a_{2,1}a_{1,4} + \frac{1}{4}a_{1,2}a_{2,2} \\ & + \frac{2}{9}a_{1,2}a_{2,3} + \frac{1}{6}a_{1,2}a_{2,4} + \frac{2}{9}a_{2,2}a_{1,3} + \frac{1}{3}a_{2,2}a_{1,4} + \frac{1}{4}a_{1,3}a_{2,3} + \frac{1}{6}a_{1,3}a_{2,4} + \frac{1}{3}a_{2,3}a_{1,4} + \frac{1}{4}a_{1,4}a_{2,4} \\ S_{1,2} = {} & \frac{1}{4}a_{1,1}a_{2,1} + \frac{1}{3}a_{1,1}a_{2,2} + \frac{1}{3}a_{1,1}a_{2,3} + \frac{4}{9}a_{1,1}a_{2,4} + \frac{1}{6}a_{2,1}a_{1,2} + \frac{1}{6}a_{2,1}a_{1,3} + \frac{1}{9}a_{2,1}a_{1,4} + \frac{1}{4}a_{1,2}a_{2,2} \\ & + \frac{2}{9}a_{1,2}a_{2,3} + \frac{1}{3}a_{1,2}a_{2,4} + \frac{2}{9}a_{2,2}a_{1,3} + \frac{1}{6}a_{2,2}a_{1,4} + \frac{1}{4}a_{1,3}a_{2,3} + \frac{1}{3}a_{1,3}a_{2,4} + \frac{1}{6}a_{2,3}a_{1,4} + \frac{1}{4}a_{1,4}a_{2,4} \\ S_{2,2} = {} & \frac{1}{4}a_{2,1}^2 + \frac{1}{2}a_{2,1}a_{2,2} + \frac{1}{2}a_{2,1}a_{2,3} + \frac{5}{9}a_{2,1}a_{2,4} + \frac{1}{4}a_{2,2}^2 + \frac{4}{9}a_{2,2}a_{2,3} + \frac{1}{2}a_{2,2}a_{2,4} \\ & + \frac{1}{4}a_{2,3}^2 + \frac{1}{2}a_{2,3}a_{2,4} + \frac{1}{4}a_{2,4}^2 \end{aligned}$$

and we can show $\mathcal{M}_{2,(2,2)} = \mathbb{C}^4$

Dimensions

Theorem [Lotter, S]**.** For $mn \leq d$ the dimension of $\mathcal{M}_{d,(m,n)}$ is

- $d\,mn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{7}{2}mn + 4(m+n) - 4$
  if m, n even,
- $d\,mn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{3}{2}mn + m + n$
  if m even and n odd,
- $d\,mn - \frac{1}{2}m^2n^2 + m^2(n-1) + (m-1)n^2 - \frac{7}{2}mn + 3(m+n) - 2$
  if m, n odd.

Example.

$$\left(\dim \mathcal{M}_{8,(m,n)}\right)_{\substack{1\leq m\leq 8\\ 1\leq n\leq 8}} = \begin{pmatrix} 8 & 15 & 21 & 26 & 30 & 33 & 35 & 36 \\ 15 & 30 & 43 & 52 & \mathbf{60} & \mathbf{62} & \mathbf{64} & 64 \\ 21 & 43 & \mathbf{52} & \mathbf{63} & \mathbf{63} & \mathbf{64} & \mathbf{64} & 64 \\ 26 & 52 & \mathbf{63} & \mathbf{63} & \mathbf{64} & \mathbf{64} & \mathbf{64} & 64 \\ 30 & \mathbf{60} & \mathbf{63} & \mathbf{64} & \mathbf{64} & \mathbf{64} & \mathbf{64} & 64 \\ 33 & \mathbf{62} & \mathbf{64} & \mathbf{64} & \mathbf{64} & \mathbf{64} & \mathbf{64} & 64 \\ 35 & \mathbf{64} & \mathbf{64} & \mathbf{64} & \mathbf{64} & \mathbf{64} & \mathbf{64} & 64 \\ 36 & 64 & 64 & 64 & 64 & 64 & 64 & 64 \end{pmatrix}$$

Outlook / questions / future work

- $\dim(\mathcal{M}_{d,(m,n)})$ unknown when $mn > d \wedge (m,n) < (d,d)$.

  Conjecture. $\dim(\mathcal{M}_{d,(m,n)}) = d^2$ if $m + n > d \wedge m \neq 1 \neq n$.

- same story for *higher tensors*, e.g. the 3-tensor of $\mathrm{Mom}^{2,2}$ is

$$\left(\begin{array}{cccc|cccc|cccc|cccc}
\frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{24} & \frac{2}{45} & \frac{1}{24} & \frac{2}{45} & \frac{1}{24} & \frac{1}{24} & \frac{2}{45} & \frac{2}{45} & \frac{1}{16} & \frac{1}{15} & \frac{1}{15} & \frac{16}{225} \\
\frac{1}{72} & \frac{1}{60} & \frac{1}{72} & \frac{1}{60} & \frac{1}{45} & \frac{1}{36} & \frac{1}{45} & \frac{1}{36} & \frac{1}{48} & \frac{1}{40} & \frac{1}{45} & \frac{2}{75} & \frac{1}{30} & \frac{1}{24} & \frac{8}{225} & \frac{2}{45} \\
\frac{1}{72} & \frac{1}{72} & \frac{1}{60} & \frac{1}{60} & \frac{1}{48} & \frac{1}{45} & \frac{1}{40} & \frac{2}{75} & \frac{1}{45} & \frac{1}{45} & \frac{1}{36} & \frac{1}{36} & \frac{1}{30} & \frac{8}{225} & \frac{1}{24} & \frac{2}{45} \\
\frac{1}{144} & \frac{1}{120} & \frac{1}{120} & \frac{1}{100} & \frac{1}{90} & \frac{1}{72} & \frac{1}{75} & \frac{1}{60} & \frac{1}{90} & \frac{1}{75} & \frac{1}{72} & \frac{1}{60} & \frac{4}{225} & \frac{1}{45} & \frac{1}{45} & \frac{1}{36}
\end{array}\right)$$

  Conjecture. No algebraic relations in the k-level *id-signature*.

- *Universal varieties* and 2-*parameter shuffles* in the *full signature* via *matrix composition Hopf algebras*?

arxiv: https://arxiv.org/abs/2409.11996

recorded talk by Felix Lotter:
https://www.math.ntnu.no/acpms/view_talk.html?id=171

Thank you!